	Final	
Name:	Department:	GRADE
Student No:	Course: Calculus II	
Signature:	Date: 06/06/2018	

Solve only 5 of the 6 problems.

1. Find the values of x for which the series converges (a) absolutely and (b) conditionally. $\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2+6}}$.

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{x^{n+1}/\sqrt{(n+1)^2+6}}{x^n/\sqrt{n^2+6}} \right| = |x| \sqrt{\lim_{n\to\infty} \frac{n^2+2n+7}{n^2+6}} = |x|$$

The series converge absolutely for $-1<x<1$.

At $x=-1$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2+6}}$, $a_n = \frac{1}{\sqrt{n^2+6}}$, i) $a_n \geq 0$, ii) a_n is decreasing. iii) $\lim_{n\to\infty} a_n = 0$

So the series converge at $x=-1$ by the Alternating series test.

At $x=1$: $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+6}}$. $\lim_{n\to\infty} \frac{1/\sqrt{n^2+6}}{1/n} = \sqrt{\lim_{n\to\infty} \frac{n^2}{n^2+6}} = 1.$

Since $\sum \frac{1}{n}$ diverges (Harmonic series), the series diverges at $x=1$ by limit comparison test.

The series converge absolutely for $-1<x<1$. conditionally at $x=-1$

2. Find an equation of the tangent plane to the sphere $x^2+y^2+z^2=6$ at the point $(1,-1,2)$.

$f = x^2+y^2+z^2-6$

$\nabla f = 2x\,i + 2y\,j + 2z\,k$

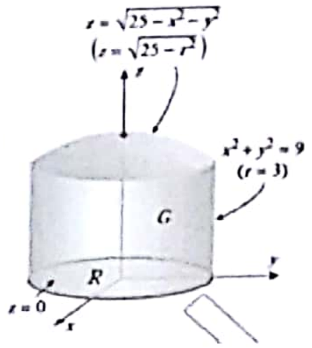
$\vec{n} = \nabla f(1,-1,2) = 2i - 2j + 4k$

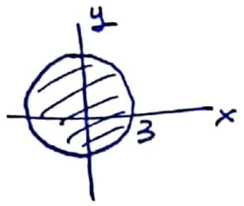
$\vec{n} \cdot \left((x-1)i + (y+1)j + (z-2)k\right) = 0$

$$\boxed{2(x-1)-2(y+1)+4(z-2)=0}$$

$x - y + 2z = 6$

3. Use triple integral in cylindrical coordinates to find the volume of the solid G that is bounded above by the semi-sphere $z = \sqrt{25 - x^2 - y^2}$, below by the xy-plane, and laterally by the cylinder $x^2 + y^2 = 9$.





$0 \leq r \leq 3$

$0 \leq \theta \leq 2\pi$

$0 \leq z \leq \sqrt{25-r^2}$

$$\text{Volume} = \int_{r=0}^{3} \int_{\theta=0}^{2\pi} \int_{z=0}^{\sqrt{25-r^2}} 1\, r\,dz\,d\theta\,dr = \int_{r=0}^{3} \int_{\theta=0}^{2\pi} \sqrt{25-r^2}\, r\,d\theta\,dr$$

$$= 2\pi \int_{r=0}^{3} \underbrace{\sqrt{25-r^2}}_{u}\, r\,dr = \pi \int_{u=25}^{16} -\sqrt{u}\,du = \pi \int_{16}^{25} \sqrt{u}\,du$$

$25 - r^2 = u$

$-2r\,dr = du$

$$= \pi \cdot \frac{2}{3} u^{3/2} \Big|_{16}^{25} = \frac{2\pi}{3}(5^3 - 4^3) = \frac{122\pi}{3}$$

$\boxed{122\pi/3}$

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4. Find the shortest distance from the origin to the surface $xyz^2 = 2$.

Take the square of distance function.

$f(x,y,z) = x^2 + y^2 + z^2$

Constraint: $g(x,y,z) = xyz^2 - 2$.

$$\nabla f = \lambda \nabla g \Rightarrow \left.\begin{aligned} 2x &= \lambda yz^2 \\ 2y &= \lambda xz^2 \\ 2z &= \lambda 2zxy \\ xyz^2 &= 2 \end{aligned}\right\} \left.\begin{aligned} 2x^2 &= \lambda xyz^2 \\ 2y^2 &= \lambda xyz^2 \\ z^2 &= \lambda xyz^2 \\ xyz^2 &= 2 \end{aligned}\right\} \begin{aligned} 2x^2 &= 2y^2 = z^2 \\ xyz^2 &= 2. \end{aligned}$$

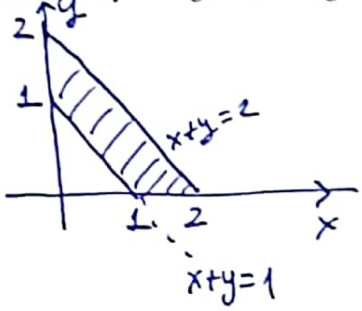
$x = y$ or $x = -y \hookrightarrow -x^2z^2 = 2$ No solution.

$x = y \hookrightarrow \left.\begin{aligned} z^2 &= 2x^2 \\ x^2z^2 &= 2 \end{aligned}\right\} \Rightarrow x^4 = 1 \Rightarrow x = \mp 1.$

Solutions: $(-1,-1,-\sqrt{2}),\ (-1,-1,\sqrt{2}),\ (1,1,-\sqrt{2}),\ (1,1,\sqrt{2})$ → distance of these points to the origin is all equal to $\sqrt{1+1+2} = 2$

$\boxed{2}$

5. Evaluate the integral $\iint_R \cos\left(\frac{y-x}{y+x}\right) dA$ where R is the trapezoidal region with vertices (1,0), (2,0), (0,2) and (0,1) by using the change of variables $u = y - x$, $v = y + x$.



$$x = \frac{v-u}{2} \qquad y = \frac{u+v}{2}$$

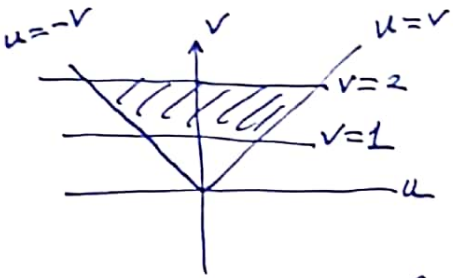
$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}.$$

$x=0 \Rightarrow u=v$

$y=0 \Rightarrow u=-v$

$x+y=1 \Rightarrow v=1$

$x+y=2 \Rightarrow v=2$



$1 \leq v \leq 2$

$-\cancel{v} \leq u \leq v$

$$\text{Integral} = \int_{v=1}^{2} \int_{u=-v}^{v} \cos\left(\frac{u}{v}\right) \frac{1}{2}\, du\, dv = \int_{v=1}^{2} \frac{1}{2} v \sin\left(\frac{u}{v}\right)\Big|_{u=-v}^{v} dv = \frac{\sin 1 - \sin(-1)}{2} \int_{v=1}^{2} v\, dv$$

$$= \frac{3 \sin 1}{2}$$

$\boxed{3\sin 1/2}$

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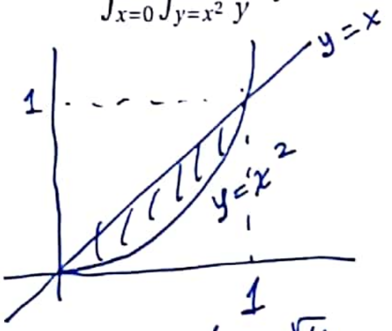
6. Evaluate $\int_{x=0}^{1} \int_{y=x^2}^{x} \frac{x}{y} dy dx$. Hint: Reverse the order of integration.



$0 \leq y \leq 1$

$y \leq x \leq \sqrt{y}$

$$\text{Integral} = \int_{y=0}^{1} \int_{x=y}^{\sqrt{y}} \frac{x}{y}\, dx\, dy = \int_{y=0}^{1} \frac{1}{y} \frac{x^2}{2}\Big|_{x=y}^{\sqrt{y}} dy$$

$$= \frac{1}{2} \int_0^1 \frac{1}{y}(y - y^2)\, dy = \frac{1}{2} \int_0^1 (1-y)\, dy = \frac{1}{2}\left(y - \frac{y^2}{2}\right)\Big|_0^1 = \frac{1}{4}$$

$\boxed{1/4}$